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Quantization from Motion Equations

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All existing quantization methods require a Lagrangian (or Hamiltonian) formulation for classical systems, while many classical systems do not possess Lagrangian descriptions (Helmholtz condition) and cannot be quantized by existing quantization methods. To make more classical systems quantizable, a new quantization procedure is proposed that skips the intermediate steps of Lagrangian description and starts directly from classical motion equations.

1. INTRODUCTION

Up to now, all quantization procedures, including canonical quantization, path integral quantization, and geometric quantization (which is essentially the globalization of the canonical one) have had to start with a Lagrangian (or Hamiltonian) description of classical systems. On the other hand, a classical system is primarily described by a set of differential equations (motion equations). In order to quantize a classical system described by a set of motion equations, all existing quantization methods have had first to establish a Lagrangian (or Hamiltonian) formulation for the system and then can go forward to perform quantization.

Some points need to be emphasized:

1. Not any given set of motion equations describing a meaningful classical system can be described by some Lagrangians (or Hamiltonians). The conditions for the existence of a Lagrangian description of a classical system are commonly called Helmholtz conditions, which have long been investigated (Helmholtz, 1887; Douglas, 1941; Currie and Saletan, 1966; Havas, 1973; Santilli, 1978; Sarlet, 1982; Pardo, 1989). Under the circumstance that no Lagrangian exists for a given classical system, no

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existing quantization methods can be used to get the corresponding quantum theory of the system. In order to make more classical systems quantizable, we are motivated to search for a new quantization procedure which does not depend on Lagrangian (or Hamiltonian) formulations.

2. When there does exist some Lagrangian description for a given classical system, there generally exist many equivalent Lagrangians that lead to the same set of classical motion equations but different quantum mechanics. This kind of ambiguity in conventional quantizations has been discussed by many workers (Hojman and Harleston, 1981; Henneaux, 1982a, b; Hojman and Shepley, 1991). In order to avoid such ambiguity, we are motivated again to establish a new quantization procedure that skips the intermediate step of Lagrangian description and starts directly from the primary motion equations.

Historically, Feynman's path integral quantization could be regarded as the first successful attemption to reduce the intermediate steps of the previous quantization method. The original quantization is the conventional canonical one. According to the canonical quantization procedure, to quantize a classical system given by a set of motion equations, one has to first find a suitable Lagrangian for the system, and then transfer to the Hamiltonian formulation through an appropriate Legendre transformation. Only after this step can the conventional canonical quantization be applied to get the corresponding quantum mechanics. As is well known, the well-definedness of the Legendre transformation is doubtful under many general situations such as constrained systems, and as a result, the conventional canonical quantization proved to be troublesome for those cases. Feynman's path integration skips the intermediate step of the Hamiltonian formulation of classical systems and starts directly from the Lagrangian formulation. Feynman succeeded not only in the field of applications, but also in providing deeper understanding of quantization.

Given the points just mentioned and the appreciation of Feynman's success, we are strongly stimulated to find a quantization method that skips not only the intermediate step of the Hamiltonian formulation, but also the step of the Lagrangian formulation. In other words, we will try to establish a new quantization method that starts directly from classical motion equations [e.g., in addition to Feynman, Wigner (1950), Okubo (1980), and Dyson (1990) also showed interest in this aspect of quantization].

In order to establish such a new quantization method, we will investigate the existing quantization rules, which are applicable for limited cases as indicated previously, and then abstract some essential rules that can be used most generally.

This paper is basically a constructive one and examines the domain of existing quantization formulations. Therefore, under the circumstance that the logical and physical pictures behind the constructions are reasonable, whether the final results are correct or true for describing nature has to be determined experimentally.

We begin with a careful review of existing quantization prescriptions.

2. REVIEW OF EXISTING QUANTIZATION METHODS

Since it is most concise to proceed with geometric terminology, let us begin with geometric quantization, which is essentially the globalization of the conventional canonical quantization procedure (Kostant, 1970; Blattner, 1977; Sternberg and Wolf, 1978; Woodhouse, 1980; Sniatycki, 1980).

A regular classical system is described by a set of second-order differential equations as follows:

$$\ddot{q}_a = f_a(q, \dot{q}), \qquad a = 1, 2, \ldots, n \tag{2.1}$$

or geometrically by a vector field T defined over phase space TQ (Q is the configuration space of the system and TQ is tangent bundle of Q):

$$T = \dot{q}_a \frac{\partial}{\partial q_a} + f_a(q, \dot{q}) \frac{\partial}{\partial \dot{q}_a} \tag{2.2}$$

For convenience, we will use the notation $v_a = \dot{q}_a$ and equations (2.1) and (2.2) can be rewritten as

$$\dot{v}_a = f_a(q, v) \tag{2.1a}$$

$$T = v_a \frac{\partial}{\partial q_a} + f_a(q, v) \frac{\partial}{\partial v_a} \tag{2.2a}$$

The Lagrangian formulation of classical systems is essentially to find a symplectic structure ω_L such that the vector field T is a Hamiltonian vector field under the symplectic structure ω_L, namely,

$$\mathscr{L}_T \omega_L = 0 \tag{2.3}$$

For some given T, there exists no such symplectic form ω_L obeying (2.3), which means the given classical system possesses no Lagrangian description on the phase space TQ. This fact is related to the Helmholtz conditions that have been investigated by many authors (Helmholtz, 1887; Douglas, 1941; Currie and Saletan, 1966; Havas, 1973; Santilli, 1978;

Sarlet, 1982; Pardo, 1989). On the other hand, for some vector field T, there may exist many different symplectic structures $\omega, \omega', \ldots$, all of them satisfying (2.3). This means the given classical system has many different but classically equivalent Lagrangian descriptions (Hojman and Harleston, 1981; Henneaux, 1982a,b; Hojman and Shepley, 1991).

The Hamiltonian formulation of classical systems can be obtained by making a Legendre map,

$$\rho_L: \quad TQ \to T^*Q \tag{2.4}$$

such that the symplectic form ω_L on TQ is mapped to the canonical symplectic 2-form ω on T^*Q and the vector field T on TQ is mapped to $\tilde{T} = \rho_L(T)$ on T^*Q.

For regular systems, it has been proved by Abraham and Marsden (1978) that the above Legendre transformation is a local diffeomorphism and therefore, for regular systems, the Hamiltonian formulation is at least locally equivalent to the Lagrangian formulation. [If the above statement is also globally valid, the systems are generally called hyperregular (Abraham and Marsden, 1978).]

Under the conditions that a given classical system possesses a Lagrangian description and the Legendre transformation has been smoothly applied, the system has a Hamilton formulation (T^*Q, H), where the vector field $\tilde{T}$ representing the flows in T^*Q is a Hamiltonian vector field of a function H under the canonical symplectic structure ω of T^*Q.

In order to quantize the system, geometric quantization first constructs a Hermite line bundle E with connection over the base manifold T^*Q. The set of square-integrable smooth sections ψ of the bundle E would be taken as some Hilbert space $\mathscr{H}$ for the corresponding quantum system. Then, by appropriately lifting to the bundle the vector field $\tilde{T} \in T(T^*Q)$ as $\tilde{T}^\uparrow \in T(E)$ which preserves the connection on the bundle, the evolution of quantum states (equivalent to the smooth sections of the bundle E) is then defined as

$$\dot{\psi} = \tilde{T}^\uparrow \psi \tag{2.5}$$

Note that the vector field $\tilde{T}^\uparrow$ preserving connection on bundle E is not the horizontal lifting of $\tilde{T}$. In fact,

$$\tilde{T}^\uparrow = \mathrm{hor}(\tilde{T}^\uparrow) + \mathrm{ver}(\tilde{T}^\uparrow) \tag{2.6}$$

the horizontal part of $\tilde{T}^\uparrow$ is the horizontal lifting of $\tilde{T}$, while the vertical part of $\tilde{T}^\uparrow$ is to be determined by the connection-preserving condition as follows:

$$\mathscr{L}_{\tilde{T}^\uparrow} A = 0 \tag{2.7}$$

where A represents the connection 1-form on bundle E.

At this stage, we have only completed the prequantization procedure in geometric quantization, since the smooth sections ψ generally depend simultaneously on coordinates and momenta, which disobey the well-accepted Heisenberg uncertainty principle and cannot be directly used as quantum wave functions.

To resolve this problem, one has to reduce the dependent variables by half. In geometric quantization, this is achieved by introducing some polarization structure P on the T^*Q, and only those polarized sections ϕ obeying

$$\nabla_X \phi = 0, \qquad \forall X \in V(T^*Q, P) \tag{2.8}$$

are chosen to construct the quantum Hilbert space $\mathscr{H}_p$.

If $\tilde{T}$ is a vector field preserving polarization P, i.e., $\mathscr{L}_{\tilde{T}} P = 0$, then $\tilde{T}^{\uparrow}\phi$ remains a polarized section and the evolution of the quantum state is still defined by

$$\dot{\phi} = \tilde{T}^{\uparrow}\phi \tag{2.9}$$

If $\tilde{T}$ does not preserve polarization P, then $\tilde{T}^{\uparrow}\phi$ will no longer be a polarized section and does not belong to the $\mathscr{H}_p$. In this case, some appropriate map (also called a pairing) ρ has to be defined to pull $\tilde{T}^{\uparrow}\phi$ back to the $\mathscr{H}_p$:

$$\rho: \quad \tilde{T}^{\uparrow}\phi \to \tilde{\phi} \in \mathscr{H}_p \tag{2.10}$$

The evolution of quantum states will be given as

$$\dot{\phi} = \tilde{\phi} = \rho(\tilde{T}^{\uparrow}\phi) \tag{2.11}$$

We have outlined the geometric quantization. The explicit expressions and some subtle mathematical structures concerning the well-definedness of the above formal equations can be found in several comprehensive books (Kostant, 1970; Blattner, Sternberg and Wolf, 1978; Woodhouse, 1980; Sniatycki, 1980).

As indicated in the Introduction, our goal of reviewing existing quantization prescriptions is to modify them and propose a new scheme of quantization that can be applied to more general situations. So, let us begin our careful examination on the geometric quantization introduced above.

There are several key rules in geometric quantization that are essentially artificial: the first one is the requirement that the vector field $\tilde{T}^{\uparrow}$ lifted from $\tilde{T}$ should preserve the connection on the bundle E; the second one is that the evolution of quantum states is determined by (2.5); the third one

is the introducing of a polarization mechanism to make the whole theory coincide with the Heisenberg principle.

Now let us try to seek out the meaning and background of these rules to see whether they are well supported or logically reasonable.

As mentioned previously, geometric quantization begins with the Hamilton formulation of classical systems; the vector field $\tilde{T}$ is the Hamilton vector field of some Hamiltonian and satisfies $\mathscr{L}_{\tilde{T}}\omega = 0$, which means $\tilde{T}$ preserves the canonical symplectic structure of T^*Q. As we will see, the requirement that $\tilde{T}^{\uparrow}$ should preserve the connection on the bundle is essentially related with the Dirac quantization condition which plays a key role in geometric quantization. The condition states that

$$\text{if} \quad f \to \hat{f} \quad \text{and} \quad g \to \hat{g}, \quad \text{then} \quad \{f, g\} \to i\hbar[\hat{f}, \hat{g}]$$

where $\hat{f}$ means the operator corresponding to classical function f, $\{\cdot, \cdot\}$ is the Poisson bracket, and $[\cdot, \cdot]$ is the commutator.

According to the condition, the curvature form of the bundle has to be proportional to the symplectic form on T^*Q. As a result, $\mathscr{L}_{\tilde{T}^{\uparrow}} A = 0$ results naturally from $\mathscr{L}_{\tilde{T}}\omega = 0$ if the Dirac quantization condition is to be obeyed [see Woodhouse (1980) for more details].

However, as pointed out by many authors, the Dirac quantization condition cannot be generally valid. (In many cases, the condition will lead to contradictions.) In fact, the quantum operators derived from geometric quantization themselves generally do not obey Dirac quantization condition (Woodhouse, 1980; Bao and Zhu, 1992a,b). With these observations in mind, why does one need at the beginning a condition that one never expects to hold afterward? [I would like to point out that in his original work Dirac (1964) did not emphasize the condition named after him.] Furthermore, when $\tilde{T}$ has no Hamilton description, namely, $\mathscr{L}_{\tilde{T}}\omega \neq 0$, there will be no solution for $\tilde{T}^{\uparrow}$ from (2.7) at all. In this case, geometric quantization cannot be performed further to get any result.

Therefore, we naturally argue that the first rule used in geometric quantization, which is closely related to the Dirac quantization condition, should be modified somewhat to allow more classical systems to be quantizable. What should the new rule be like? In other words, what new condition should be adopted to determine the vector field $\tilde{T}^{\uparrow}$ to be lifted from $\tilde{T}$? An answer to this question is a constructive one and will be given in the next section.

As for the second rule given by (2.5), it is related to the Copenhagen interpretation of quantum mechanics, which requires that the quantum evolution equation be first-order in the time derivative (Bohm, 1954). We will keep the Copenhagen interpretation and so we will maintain this rule in the forthcoming constructions of a new quantization procedure.

Now, let us turn to the third rule—the introduction of polarization and the pairing mechanism. As one can see, the only purpose of these procedures is to make the theory obey the Heisenberg uncertainty principle, while the main strategy is to select those polarized sections as physical wave functions. However, in the author's opinion, this method adopted by geometric quantization is not absolutely necessary. Actually, in order to coincide with the Heisenberg principle, the physical wave functions can be found simply by integrating out momentum variables in the functions $\psi(q, p)$ corresponding to the smooth sections of bundle E over T^*Q. Namely,

$$\phi(q) = \int dp \cdot \psi(q, p) \tag{2.12}$$

Obviously, there exist many functions $\psi(q, p)$ whose integration over p leads to the same result $\phi(q)$. We will denote the set of these functions as

$$\mathscr{F}_\phi = \left\{ \psi(q, p) \middle| \int dp \cdot \psi(q, p) = \phi(q) \right\} \tag{2.13}$$

It seems to the author that, in the Schrödinger representation, it is much more convenient to adopt this simple prescription and avoid using the complicated polarization mechanism.

As a conclusion, we have examined existing quantization methods mainly in the geometric framework. Some artificial rules behind the quantization method have been carefully discussed and possible modifications have been suggested. In the next section, we will try to construct a new quantization prescription with the ideas developed in this section, which is intended to be applicable to more general classical systems including those without a Lagrangian formulation.

3. CONSTRUCTION OF NEW QUANTIZATION PROCEDURE

As mentioned previously, a general classical system is primarily described by a set of motion equations. Whether the system possesses a Lagrangian (or Hamiltonian) formulation is not guaranteed, but is subject to the Helmholtz conditions. In order to quantize those systems disobeying the Helmholtz conditions, one need a new quantization procedure that can skip the intermediate processes and involve only the original motion equations.

To be concrete, we will still concentrate on regular systems described by the following motion equations:

$$\dot{v}_a = f_a(q, v) \tag{3.1}$$

or equivalently by the vector field T over the phase space TQ,

$$T = v_a \frac{\partial}{\partial q_a} + f_a(q, v) \frac{\partial}{\partial v_a} \tag{3.2}$$

With the ideas developed in the last section, the quantum evolution equation of physical wave functions $\phi(q)$ can be constructed as follows:

First, considering the Heisenberg principle, we employ (2.14) and express physical wave functions as

$$\phi(q) = \int d^n v_a \cdot \psi(q, v) \tag{3.3}$$

where n is the dimension of configuration space Q.

The time evolution of $\phi(q)$ is obviously

$$\dot{\phi}(q) = \int d^n v_a \cdot \dot{\psi}(q, v) \tag{3.4}$$

Then we make use of (2.5), which is to be maintained as discussed in the previous section,

$$\dot{\psi}(q, v) = T^{\uparrow} \psi(q, v) \tag{3.5}$$

where $T^{\uparrow}$ is some appropriate vector field lifted from the original T. Since we have argued that the Dirac condition has to be given up, the vertical part of $T^{\uparrow}$ is unknown and will be determined by some other way to be investigated later.

According to equations (3.4) and (3.5), we can have the following formal equation for the quantum evolution of physical wavefunctions:

$$\dot{\phi}(q) = \int d^n v_a \cdot T^{\uparrow} \psi(q, v) \tag{3.6}$$

where

$$\psi \in \mathscr{F}_{\phi} = \left\{ \psi(q, v) \,\middle|\, d^n v_a \cdot \psi(q, v) = \phi(q) \right\} \tag{3.7}$$

Now, before we undertake any concrete construction of $T^{\uparrow}$, let us note that an important distinction exists between nonrelativistic classical description of dynamical systems and the quantum version. To see this point, we discuss a simple case of a conservative system.

The classical motion equations for the system are

$$\dot{v}_a = -\frac{\partial}{\partial q_a} V(q) \tag{3.8}$$

which may be described equivalently as

$$m_1 \dot{v}_a = -\frac{\partial}{\partial q_a} V_1(q) \qquad \text{with} \quad V_1(q) = m_1 V(q) \tag{3.9a}$$

or

$$m_2 \dot{v}_a = -\frac{\partial}{\partial q_a} V_2(q) \qquad \text{with} \quad V_2(q) = m_2 V(q) \tag{3.9b}$$

Therefore, from the motion equations (3.8), we cannot distinguish the cases (I) that the mass of the particle is $_1$ and the conservative potential is $V_1(q)$, or (II) the mass of the particle is m_2 and the conservative potential is $V_2(q)$.

However, in quantum mechanics, the situation is quite different. The Schrödinger equation for case I is

$$i\hbar\dot{\psi} = \left(\frac{-\hbar^2\nabla^2}{2m_1} + V_1(q)\right)\psi \tag{3.10a}$$

and for case II is

$$i\hbar\dot{\psi} = \left(\frac{-\hbar^2\nabla^2}{2m_2} + V_2\right)\psi = \frac{m_1}{m_2}\left(\frac{-\hbar^2\nabla^2}{2m_1} + \frac{m_2^2}{m_1^2} V_1\right)\psi \tag{3.10b}$$

In contrast to the fact that (3.9a) and (3.9b) are dynamically equivalent, equations (3.10a) and (3.10b) have different quantum dynamical behaviors. In other words, cases I and II are distinguishable in quantum theory.

The key point of why nonrelativistic classical mechanics (excluding gravity) differs from quantum mechanics is that there exists a universal constant $\hbar$ in quantum theory which disallows any rescaling of the mass of particles. Similar situations occur in relativistic theory and gravity theory, where other universal constants, c (speed of light) and G (gravitational

constant), exist. Therefore, in these theories, the mass must be considered as a definite property of a particle, which cannot be mixed up with environmental factors. As a result, for nonrelativistic dynamics, the mass of a particle seems to have to be put into the quantization procedure by hand. The question is, at what stage of quantization should we insert the concept of the mass of a particle?

Let us first notice that equations (3.10a) and (3.10b) can be written in the following unified form:

$$i\hbar_m \dot{\psi} = \left[\left(\frac{-\hbar_m^2}{2} \right) \nabla^2 + V(q) \right] \cdot \psi \tag{3.11}$$

where $V(q)$ is the function given in the classical motion equations (3.8) and $\hbar_m = \hbar/m$. By substituting $\hbar_{m_1} = \hbar/m_1$ and $\hbar_{m_2} = \hbar/m_2$ into (3.11), one can easily regain equations (3.10a) and (3.10b).

The above fact may be regarded as just a formal transformation. But a more interesting interpretation could be achieved by regarding the parameter $\hbar_m$ in equation (3.11) as the "quanta" of the particle, representing the sensitivity of the particle to quantum phenomena. Different particles may have different quantum sensitivities and they can be determined through experiments.

In this way, we actually do not need the concept of mass beforehand in the theoretical structure of quantum theory. Alternatively, the concept of mass now turns out as a consequent one. That is, we may define the quantity $\hbar_m/\hbar$, which represents the quantum sensitivity of a particle, to be a kind of mass m_Q that would be best called "nonrelativistic quantum mechanical mass."

That this quantity m_Q is equivalent to the gravitational mass m_G defined in gravity theory is not guaranteed and is a profound problem to be settled by further theoretical considerations and experiments. We discuss it elsewhere. In the following, we will only keep in mind the concept of the quantum sensitivity of a particle which is represented by a quantity $\hbar_m$ to be determined by experiments.

Now, let us return to continue our construction of the new quantization procedure. The quantum evolution equation of physical wave functions has been formally given in (3.6). In order for it to be expressed in familiar Schrödinger-like form, we would like to select an appropriate function ψ in the set $\mathscr{F}_\phi$. Inspired by the famous Wigner (1963, 1983) transformation

$$\psi(q) \to W(q, p) = (1/2\pi\hbar) \int dq' \exp(ipq'/\hbar) \cdot \psi^*(q + q'/2)\, \psi(q - q'/2) \tag{3.12}$$

we adopt the following similar choice:

$$\psi(q, v) = \exp(iq_a v_a/\hbar_m) \cdot \bar{\phi}(v) \tag{3.13}$$

where $\hbar_m$ is the quantity defined previously and $\bar{\phi}(v)$ is the Fourier transformation of $\phi(q)$,

$$\bar{\phi}(v) = (2\pi/\hbar_m)^n \int d^n q'_a \cdot \phi(q') \exp(-iv_b v'_b/\hbar_m) \tag{3.14}$$

One can easily prove that the above $\psi(q, v)$ is a function in the set $\mathscr{F}_\phi$ corresponding to the physical wave function ϕ. Actually,

$$\int d^n v_a \cdot \psi(q, v) = (2\pi/\hbar_m)^n \int d^n v_a \, d^n q'_b \cdot \phi(q') \exp[iv_a(q_a - q'_a)/\hbar_m]$$

$$= \int d^n q_a \cdot \phi(q') \cdot \delta(q_a - q'_a) = \phi(q) \tag{3.15}$$

Now, we will determine the vector field $T^\uparrow$ which represents the essential dynamics of the quantum world. Like many events in the history of scientific research, there seems no first principle that can give us the exact form of $T^\uparrow$. The best one can do is to make some logical postulation which will lead to reasonable results comparable with experiments. After several trials, we think that the following construction for $T^\uparrow$ might be an acceptable one:

$$T^\uparrow = \frac{-1}{2} \cdot (i_T \, dq_a) \frac{\partial}{\partial q_a} + \frac{i}{\hbar_m} \int d^n q_a \cdot (i_T \, dv_a) \tag{3.16}$$

where i_T is the inner product associated with the original vector field T. This form of $T^\uparrow$ is adopted since it is very simple and will give the well-accepted Schrödinger equation for conservative systems. (See Example 1 in the next section.)

As a result, we have established a new quantization procedure that skips the Lagrangian formulation. With combined use of equations (3.16), (3.13), and (3.6), any regular classical system given by (3.1) can be quantized, no matter whether it possesses a Lagrangian or a Hamiltonian formulation. In the next section, we will demonstrate the new quantization procedure proposed in this section with nontrivial examples.

4. EXAMPLES AND DISCUSSIONS

This section is devoted to two examples. The first example is about conservative systems. Actually, the general form of the vector field $T^\uparrow$ given

in the last section in (3.16) is proposed in order to regain the well-accepted Schrödinger equation for conservative systems. The second example will discuss a classical system that has no Lagrangian formulation and cannot be quantized via previous quantization methods.

Example 1. Let us consider a conservative system described by the following motion equation:

$$\ddot{q} = f(q) = -\frac{\partial V(q)}{\partial q} \tag{4.1}$$

In vector field form, the systems are equivalently described by

$$T = v\frac{\partial}{\partial q} + f(q)\frac{\partial}{\partial v} \tag{4.2}$$

According to (3.16), the lifted vector field $T^{\uparrow}$ can be evaluated as

$$T^{\uparrow} = \frac{-1}{2} v\frac{\partial}{\partial q} - \frac{i}{\hbar_m} V(q) \tag{4.3}$$

Henceforth, the quantum evolution for the systems can be derived from (3.6) and (4.3) as follows:

$$\dot{\phi} = \int dv \cdot T^{\uparrow}\psi(q, v) = \frac{-1}{2}\int dv \cdot v\frac{\partial \psi}{\partial q} - \frac{i}{\hbar_m} V(q)\,\phi(q) \tag{4.4}$$

Now, substituting the expansion (3.13) into the above equation, we will get

$$\begin{aligned}\int dv \cdot v\frac{\partial \psi}{\partial q} &= \int dv \cdot v\frac{i}{\hbar_m} v \exp\left(\frac{iqv}{\hbar_m}\right)\bar{\phi}(v) \\ &= \frac{\hbar_m}{i}\frac{\partial^2}{\partial q^2}\int dv \cdot \exp\left(\frac{iqv}{\hbar_m}\right)\bar{\phi}(v) = \frac{\hbar_m}{i}\frac{\partial^2 \phi}{\partial q^2}\end{aligned} \tag{4.5}$$

As a result, equation (4.4) becomes

$$\dot{\phi} = \frac{-1}{2}\frac{\hbar_m}{i}\frac{\partial^2 \phi}{\partial q^2} - \frac{i}{\hbar_m} V(q)\,\phi(q) \tag{4.6}$$

or in the more familiar form

$$i\hbar_m \dot{\phi} = \frac{-\hbar_m^2}{2}\frac{\partial^2 \phi}{\partial q^2} - V(q)\,\phi(q) \tag{4.7}$$

which is exactly the Schrödinger equation for conservative systems. [See equation (3.11) for the mass problem.]

Example 2. Now we consider a two-dimensional classical system described by the following equations:

$$\ddot{x}+\dot{y}=0 \qquad \text{and} \qquad \ddot{y}+y=0 \tag{4.8}$$

According to Douglas (1941) and Hojman and Shepley (1991), this system does not obey the Helmholtz condition and has no Lagrangian formulation in phase space. Previous quantization methods have to regard this system as nonquantizable. However, we will see how this system can be quantized under our new quantization procedure.

For convenience, let us adopt the following notations:

$$q_1=x, \qquad q_2=y, \qquad v_1=\dot{x}, \qquad v_2=\dot{y}$$

The vector field T representing the system is

$$T=\left(v_1\frac{\partial}{\partial q_1}+v_2\frac{\partial}{\partial q_2}\right)-\left(v2\frac{\partial}{\partial v_1}+q_2\frac{\partial}{\partial v_2}\right) \tag{4.9}$$

Substituting (4.9) into (3.16), we can easily evaluate the lifted vector field $T^{\uparrow}$ by straightforward calculations,

$$T^{\uparrow}=\frac{-1}{2}\left(v_1\frac{\partial}{\partial q_1}+v_2\frac{\partial}{\partial q_2}\right)-\frac{i}{\hbar_m}\left(q_1v_2+\frac{q_2^2}{2}\right) \tag{4.10}$$

Now, according to (3.6) and making use of the choice (3.13), the quantum evolution equation for the system can be derived as follows:

$$i\hbar_m\dot{\phi}=-\frac{\hbar_m^2}{2}\left(\frac{\partial^2}{\partial q_1^2}+\frac{\partial^2}{\partial q_2^2}\right)\phi-i\hbar_m q_1\frac{\partial\phi}{\partial q_2}+\frac{1}{2}q_2^2\phi \tag{4.11}$$

5. CONCLUSION

As a final remark, we have proposed a new quantization procedure that skips the Lagrangian formulation and therefore makes more classical systems quantizable. Whether this method is a correct one is subject to further examination both theoretically and experimentally. Though the author believes that the formal form of the quantum evolution equation (3.6) is well supported, the concrete form of the lifted vector field $T^{\uparrow}$ in (3.16) might be possibly modified in some way, since it has been constructed only to regain the well-accepted Schrödinger equations.

Furthermore, to generalize the discussions of this paper to singular (constrained) systems will be interesting. Another important point to investigate further is the problem related with the concept of mass as indicated in Section 3.

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REFERENCES

Abraham, R., and Marsden, J. E. (1978). *Foundations of Mechanics*, 2nd ed., Benjamin/Cummings, Menlo Park, California.
Bao, Dehai, and Zhu, Z. Y. (1992a). *Journal of Physics A*, **25**, 2381.
Bao, Dehai, and Zhu, Z. Y. (1992b). On Dirac's quantization condition, CCAST-92-19.
Blattner, R. J. (1977). *Lecture Notes in Mathematics*, Vol. 570, Springer, Berlin.
Bohm, D. (1954). *Quantum Theory*, Prentice-Hall, New York.
Currie, D. D., and Saletan, E. J. (1966). *Journal of Mathematical Physics*, **7**, 967.
Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Yeshiva University, New York.
Douglas, J. (1941). *Transactions of the American Mathematical Society*, **50**, 71.
Dyson, F. J. (1990). *American Journal of Physics*, **58**, 209.
Havas, P. (1973). *Acta Physica Austriaca*, **38**, 145.
Helmholtz, H. (1887). *Journal für die Reine und Angewandte Mathematik*, **100**, 137.
Henneaux, M. (1982a). *Journal of Physics A*, **15**, L93.
Henneaux, M. (1982b). *Annals of Physics*, **140**, 1.
Hojman, S., and Harleston, H. (1981). *Journal of Mathematical Physics*, **22**, 1414.
Hojman, S. A., and Shepley, L. C. (1991). *Journal of Mathematical Physics*, **32**, 142.
Kostant, B. (1970). *Lecture Notes in Mathematics*, Vol. 170, Springer, Berlin.
Okubo, S. (1980). *Physical Review D*, **22**, 1919.
Pardo, F. (1989). *Journal of Mathematical Physics*, **30**, 2054.
Santilli, R. M. (1978). *Foundations of Theoretical Mechanics*, Springer, Berlin.
Sarlet, W. (1982). *Journal of Physics A*, **15**, 1503.
Sniatycki, J. (1980). *Geometric Quantization and Quantum Mechanics*, Springer-Verlag, Berlin.
Sternberg, S., and Wolf, J. A. (1978). *Transactions of the American Mathematical Society*, **238**.
Wigner, E. P. (1950). *Physical Review*, **77**, 711.
Wigner, E. P. (1963). *American Journal of Physics*, **31**, 6.
Wigner, E. P. (1983). In *Quantum Optics, Experimental Gravitation and the Measurement Theory*, P. Meystre, ed., Plenum Press, New York.
Woodhouse, M. (1980). *Geometric Quantization*, Oxford University Press, Oxford.